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Empirical Bayes Analysis for Exponential-Weibull Distrubition Family in Moving Extremes Ranked Set sampling design

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Abstract

In this paper, we get the empirical Bayes test rules for Exponential-Weibull distribution in moving extremes ranked set. Its asymptotic optimality and convergence rate are obtained.

Keywords

Moving extremes ranked set, empirical Bayes, Exponential distribution.

1. Introduction

Ranked set sampling (RSS) was introduced by McIntyre to estimated pasture yields[1]. Moving extremes ranked set sampling has been used in many fields such as environment and sociology. Since Robbins proposed empirical Bayes (EB) approach, it has been developed [2-8].

However, EB method are based on simple random sampling. An idea is to apply EB methods in moving extremes ranked set sampling. Recently, empirical Bayes test rule and its asymptotical property for the parameter of power distribution based on RSS has been established[9]. In this paper, we obtain empirical Bayes test rule for Exponential-Weibull distribution in moving extremes ranked set sampling.

Let X have a conditional density function for given θ

$$f(x\mid\theta)=e^{-x}\theta(1-e^{-x})^{\theta-1} \tag{1.1}$$

where θ is unknown parameterand $\Theta=\{\theta>0\}$ is parameter space.

We discuss the following test problem:

$$H_0:\theta\le\theta_0 \leftrightarrow H_1:\theta>\theta_0 \tag{1.2}$$

where $\theta 0$ is given constants.

We choose loss function $L_0(\theta,d_0)=a\frac{(\theta-\theta_0)^2}{\theta}I(\theta>\theta_0),$

$$L_1(\theta,d_1)=a\frac{(\theta-\theta_0)^2}{\theta}I(\theta\le\theta_0).$$

where a >0,d={d0,d1} is action space, d0 and d1 imply acceptance and rejection of $H0$ respectively.

Suppose that the prior distribution G(θ) of parameter the θ is unknown.

We have random decision function

$$\delta(x)=P(\text{accept } H0|X=x). \tag{1.3}$$

Then, the risk function of δ(x) is shown by

$$R(\delta(x),G(\theta))=\int_{\Theta}\int_{\Omega}\left[L_0(\theta,d_0)f(x|\theta)\delta(x)+\right.$$
$$\left.L_1(\theta,d_1)f(x|\theta)(1-\delta(x))\right]dxdG(\theta)$$
$$=a\int_{\Omega}\beta(x)\delta(x)dx+C_G$$

where

$$C_G=\int_{\Omega}L_1(\theta,d_1)dG(\theta),\ \beta(x)=\int_{\Theta}\frac{(\theta-\theta_0)^2}{\theta}f(x|\theta)dG(\theta). \tag{1.4}$$

The marginal density function of X is shown by

$$f_G(x)=\int_{\Theta}f(x|\theta)\mathrm{d}G(\theta),\beta(x)=\int_{\Theta}\mathrm{e}^{-x}(1-\mathrm{e}^{-x})^{\theta-1}\mathrm{d}G(\theta),$$

where

$$f_G^{(1)}=-f_G(x)+\int_{\Theta}\mathrm{e}^{-2x}\theta(\theta-1)(1-\mathrm{e}^{-x})^{\theta-2}\mathrm{d}G(\theta)=-f_G(x)+\frac{1}{\mathrm{e}^x-1}[\int_{\Theta}\theta f(x|\theta)\mathrm{d}G(\theta)-f_G(x)], \tag{1.5}$$

By (1.5), we have $\beta(x)=u_1(x)f_G^{(1)}(x)+u_2(x)f_G(x)$

where $u_1(x)=\mathrm{e}^{2x}-\mathrm{e}^x+3$, $u_2(x)=(\mathrm{e}^x-1)(\mathrm{e}^x-\theta_0)$.

Using (1.5), Bayes test function is obtained as follows

$$\delta_G(x)=\begin{cases}1,\beta(x)\le 0\\0,\beta(x)>0\end{cases}$$

Further, we obtian the minimum Bayes risk as follows

$$R(G)=\inf_{\delta}R(\delta,G)=R(\delta_G,G)$$
$$=a\int_{\Omega}\beta(x)\delta_G(x)dx+C_G \tag{1.6}$$

From above that δ(x)=δG(x) and R(G) can be obtained when the prior distribution of G(θ) is given. If not, we use the EB method. The rest of this paper is organized as follows. Section 2 presents an EB test under ranked set sampling. In section 3, we obtain asymptotic optimality and the optimal rate of convergence of the EB test in moving extremes ranked set sampling.

2. Construction of EB test under Moving Extremes Ranked Set

Supposed that X(1)1,X(1)2,⋯,X(1)m,X(2)1,X(2)2⋯,X(2)m,⋯,X(k)1,X(k)2⋯,X(k)m be a balanced moving extremes ranked set sample from population which has the common marginal density function fG(x). We assume perfect ranking. Denote thatX(1)1 ,X(1)m,X(2)1 ,X(2)m, X(k)1 ,X(k)m are moving extremes ranked set historical samples, and X is present sample. Assume f(x)∈C*s*,*α*,x∈ R1, where C*s*,*α*={g(x)|g(x) is a probability density function; the s−he order derivative *g*(*s*)(x) is continuous with |*g*(*s*)(x)|≤α,s≥3,α>0}, $n=km$.

Supposed that Kr(x) be a Borel measurable bounded function vanishing off (0,1) such that (C1):

$$\frac{1}{t!}\int_0^1 v^tK(v)dv=\begin{cases}1,t=0\\0,t=1,\cdots,s-1\end{cases} \tag{2.1}$$

Kernel estimator of f(x) is defined by

fn(x)=1/*mkhn*∑Kr(x−X(i)1/*hn*)+ Kr(x−X(i)m/*hn*)

where *hn* is a positive and smoothing bandwidth, and lim*n*→∞*hn*=0. Thus, the estimator of β(x) is shown by

$$\beta_n(x)=u_1(x)f_n^{(1)}(x)+u_2(x)f_n(x) \tag{2.2}$$

And, the EB test function is defined as follows

$$\delta_n(x)=\begin{cases}1,\beta_n(x)\le 0\\0,\beta_n(x)>0\end{cases} \tag{2.3}$$

Let E stand for mathematical expectation with respect to the joint distribution of X(1)1, X(1)m,X(2)1, X(2)mX(k)1, X(k)m. Then, the overall Bayes risk of δn(x) is shown by

$$R(\delta_n(x),G)=a\int_{\Omega}\beta(x)E_n[\delta_n(x)]dx+C_G$$

If lim$n$→∞ R(δn,G)=R(δG,G),{δn(x)} is called asymptotic optimality of EB test function. If $R(\delta_n,G)-R(\delta_G,G)=O(n^{-q})$, where q>0, $O(n^{-q})$ is asymptotic optimality convergence rates of EB test function {δn(x)}. Before proving the theorems, we need the following lemmas. Supposed that c,c1 be different constants in different cases even in the same expression.
Lemma. R(δG,G) and R(δn,G) are defined by above, then
0≤R(δn,G)−R(δG,G)≤a∫|β(x)|P(|βn(x)−β(x)|≥|β(x)|)dxΩ.

3. Asymptotic Optimality and Convergence Rates of Empirical Bayes test in Moving Extremes Ranked Set

Theorem 3.1. Assume (C1) and the following regularity conditions hold.
(1)hn>0, lim$n$→∞hn=0,
(2)∫θ^2 dG(θ)<∞,
(3)f (x) is continuous function,
Then, lim$n$→∞ R(δn, G)=R(δG,G).
Proof. Lemma 1 shows that 0≤R(δn,G)−R(δG,G)≤a∫|β(x)|P(|βn(x)−β(x)|≥|β(x)|)dxΩ.
Applying Fubini theorem,
we have $\int_{\Omega}|\beta(x)|)\mathrm{d}x=\int_{\Omega}\int_{\Theta}\left|(\theta_0-1)^2+(3-\theta_0)(\theta-2)+(\theta+1)(\theta-2)\right|f(x|\theta)\mathrm{d}G(\theta)\,\mathrm{d}x\leqslant$
$\int_{\Omega}\int_{\Theta}|(\theta_0-1)^2|f(x|\theta)\mathrm{d}G(\theta)\,\mathrm{d}x+\int_{\Omega}\int_{\Theta}|(3-2\theta_0)(\theta-2)|f(x|\theta)\mathrm{d}G(\theta)\,\mathrm{d}x+\int_{\Omega}\int_{\Theta}|(\theta+1)(\theta-2)|f(x|\theta)\mathrm{d}G(\theta)\,\mathrm{d}x\leqslant$
$(\theta_0-1)^2+|(3-2\theta_0)|\int_{\Theta}|\theta-1|\mathrm{d}G(\theta)+\int_{\Theta}|(\theta+1)(\theta-2)|\mathrm{d}G(\theta)<+\infty.$

Denote $\Psi_n(x)$=|β(x)|P(|βn(x)−β(x)|≥|β(x)|).

Obviously, $\Psi_n(x)$≤|β(x)|. Then, by domain convergence theorem, we have

$$0\le\lim_{n\to\infty} R(\delta n, G)-R(\delta G,G)\le\int[\lim_{n\to\infty}\Psi_n(x)]\Omega dx. \tag{3.1}$$

Next, we need prove that lim$n$→∞ $\Psi_n(x)$ =0 holds almost everywhere. By Markov's and Jensen's inequality,

$$\lim_{n\to\infty}\Psi_n(x)=0 \tag{3.2}$$

Substituting (3.2) into (3.1), the proof of theorem 3.1 is finished.
Theorem 3.2. Assume (C1) and the following regularity conditions hold.
fG(x)∈Cs,α, where $h_n=n^{-1/(2+s)}$,
(4) $\int_{\Omega}\mathrm{e}^{m\lambda x}|\beta(x)|^{1-\lambda}\,\mathrm{d}x<+\infty$, $m=0,1,2$
Then, we get $R(\delta_n,G)-R(\delta_G,G)=O\left(n^{-\lambda(s-1)/2s+1}\right)$
Proof. Using Markov's inequality, we get

$$\begin{aligned}
0 &\le R(\delta_n, G) - R(\delta_G, G) \\
&\le a\int_\Omega |\beta(x)|^{1-\lambda} E|\beta_n(x)-\beta(x)|^\lambda dx \\
&\le c\int_\Omega |\beta(x)|^{1-\lambda} E|f_n(x)-f_G(x)|^\lambda dx + \\
&\quad c\int_\Omega |\beta(x)|^{1-\lambda} m^\lambda(x) E|\phi_n(x)-\phi_G(x)|^\lambda dx \\
&\equiv A_n + B_n
\end{aligned} \tag{3.3}$$

Applying Lemma and the conditions (4) in the Theorem (3.2), we obtain

$$A_n \leqslant c_1 n^{-\frac{\lambda(s-2)}{2s+1}} \int_\Omega |\beta(x)|^{1-\lambda} |u_1(x)|^\lambda \mathrm{d}x \leqslant c_4 n^{-\frac{\lambda(s-2)}{2s+1}}, \tag{3.4}$$

$$B_n \leqslant c_2 n^{-\frac{\lambda(s-1)}{2s+1}} \int_\Omega |\beta(x)|^{1-\lambda} |u_2(x)|^\lambda \mathrm{d}x \leqslant c_5 n^{-\frac{\lambda(s-1)}{2s+1}}, \tag{3.5}$$

Substituting (3.4)-(3.5) into (3.3), we have R(δn, G)−R(δG, G)= $O\left(n^{-\lambda(s-2)/2(s+1)}\right)$.

The proof of theorem 3.2 is finished.

Remark 3.1. When $\lambda \to 1$, $s \to \infty$, $O\left(n^{-\lambda(s-1)/2s+1}\right)$ nears $O\left(n^{-1/2}\right)$.

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